Problem Sheet 5.

Among all natural numbers we can distinguish *prime* and *composite* numbers.

A number is *composite* if it is a product of two smaller natural numbers. For example, $6 = 2 \times 3$. Otherwise, and if the number is not equal to 1, it is called *prime*. The number 1 is neither prime nor composite.

The well-known fact, called the **Fundamental Theorem of Arithmetic**, says that any natural number greater than 1 can be uniquely expressed as a product of prime numbers in non-decreasing order. For example, $630 = 2 \times 3 \times 3 \times 5 \times 7 =$ $= 2 \times 3^2 \times 5 \times 7$ (the second expression is more common for prime decomposition).

Modulo operation: Given any two natural numbers a and b, called the dividend and the divisor respectively, we can divide a by b with the remainder. That is to find such non-negative integer numbers c and d ($d<b$), called the quotient and the remainder respectively, that $a = c \times b + d$. For example, $41 = 2 \times 15 + 11$ is the division of 41 by 15 with the remainder 11 and $5 = 0 \times 7 + 5$ is the division of 5 by 7 with the remainder 5.

If a is divided by b with zero remainder (without a remainder) we say that "a is divisible by b" or "b divides a". From the definition of *modulo operation* for a the property to be divisible by b is equivalent to the existence of non-negative integer c such that $a = c \times b$. We denote it by $a \,\vdots\, b$ for "a is divisible by b" and $b|a$ for "b divides a". For example, $105 \,\vdots\, 7$ and $9|111111111$ because $105 = 15 \times 7$ and $111111111 = 12345679 \times 9$.

We immediately deduce from the Fundamental Theorem of Arithmetic that if a product of two natural numbers is divisible by a prime number, then one of these numbers is divisible by this prime number.

Example 1. Is it true that if a natural number is divisible by 4 and by 6, then it must be divisible by $4 \times 6 = 24$?
Example 2. And what if a natural number is divisible by 5 and by 7? Should it be divisible by 35?
Example 3. The number A is not divisible by 3. Is it possible that the number $2A$ is divisible by 3?

Problem 5.1. List the first 25 prime numbers. Done? Now write the prime decomposition of 2910.
Problem 5.2. Lisa knows that A is an even number. But she is not sure if $3A$ is divisible by 6. What do you think?
Problem 5.3. George divided number a by number b with the remainder d and the quotient c. How will the remainder and the quotient change if the dividend and the divisor are increased by a factor of 3?

We denote the product of all natural numbers from 1 to n by $n!$. For example, $5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$.

Problem 5.4. a) Prove that the product of any three consecutive natural numbers is divisible by 3!=6. b) What about the product of any four consecutive natural numbers? Is it always divisible by 4!=24?
Problem 5.5. A young mathematician felt very sad and lonely during New Year's Eve. The main reason for his sadness (have you guessed already?) was the lack of mathematical problems. So he decided to create a new one on his own. He wrote the following words on a small piece of paper: "Find the smallest natural number n such that n! is divisible by 2016" , but unfortunately he immediately forgot the answer. What is the correct answer to this question?