Problem Sheet 7.

Recall the definition of modulo operation from Problem Sheet 5:

Modulo operation: Given any two natural numbers a and b, called the dividend and the divisor respectively, we can divide a by b with a remainder. That is to find such non-negative integer numbers c and d (with $d<b$), called the quotient and the remainder respectively, that $a = c \times b + d$. For example, $41 = 2 \times 15 + 11$ is the division of 41 by 15 with the remainder 11, and $5 = 0 \times 7 + 5$ is the division of 5 by 7 with the remainder 5.

Instead of saying "a has remainder d from division by b" we can use "a and d are congruent modulo b" and write $a \equiv d\,(mod\ b)$. In fact, the last equality we can write for any number d such that it has the same remainder as number a from division by b.

We can manipulate remainders in the same way as we do with natural numbers (meaning that we can add them up and multiply by each other). This is due to the following proposition.

Proposition. Fix the divisor n and consider the division (modulo operation) by n, then
(a) the sum of any two natural numbers has the same remainder as the sum of the remainders of these natural numbers;
(b) the product of any two natural numbers has the same remainder as the product of the remainders of these natural numbers.

PROOF: (a),(b) Let x and y be our natural numbers. Assume that $x = k \times n + r_1$ and $y = l \times n + r_2$ are the expressions obtained from the modulo operation. Then $x + y = (k + l) \times n + (r_1 + r_2)$ and $x \times y = (k \times r_2 + l \times r_1 + k \times l \times n) \times n + r_1 \times r_2$. It means $(r_1 + r_2)$ has the same remainder as $(x + y)$ from division by n, and $r_1 \times r_2$ has the same remainder as $x \times y$ from division by n.

In new notation: if $x \equiv r_1\,(mod\ n)$ and $y \equiv r_2\,(mod\ n)$, then $x + y \equiv r_1 + r_2\,(mod\ n)$ and $x \times y \equiv r_1 \times r_2\,(mod\ n)$.

Example 1. Make a table for summation and multiplication of the remainders for n=3 and n=5.

Answer:

n=3,+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

n=3,×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

n=5,+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

n=5,×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

Example 2. What is the remainder of 7^{57} from division by 5?

Example 3. What possible remainders can we get when a square number is divided by 3?

Problem 7.1. Bilbo Baggins put in his diary the tables for summation and multiplication of all remainders for $n = 7$. Can you reproduce them?

Problem 7.2. Find the remainder when the number $2015 \times 2016 \times 2017 + 2014^2 \times \times 2018^3$ is divided by 11.

Problem 7.3. There are 99 bags with gold in Smaug's cave in the Lonely Mountain. They are placed in a circle and each of them contains coins (if you count the coins in the bags you get natural numbers). The dwarves know that any two neighbouring bags differ by 1 or by 2 or double their neighbour. Is it possible that no bag contains such a number of coins, that is a multiple of 3?

Problem 7.4. The elves are good not only at archery. They also have extensive knowledge in arithmetics and number theory. Without a doubt they know the technique of remainders. Based on this theory they could even answer whether it is possible for a sum of three squares of odd natural numbers to be a perfect square. Is it possible in your opinion?

Problem 7.5. Gandalf the Grey was always missing throughout the journey. This time he was heading to visit Galadriel, Lord Elrond and Saruman. Apparently it was a really long ride, so he had a lot of time to master his wisdom. He proved that $2222^{5555} + 5555^{2222}$ is a multiple of 7. Can you prove it?